Numerical Methods I
Fourier and Wavelet Transforms

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Outline

1. Fourier Orthogonal Basis
2. Fast Fourier Transform
3. Applications of FFT
4. Wavelets
5. Conclusions

Periodic Functions

- We are considering the space $L^2_{2\pi}$ of **square-integrable periodic functions** defined on the interval $I = [0, 2\pi]$.
- The Fourier basis is a family of **orthogonal exponential functions**

$$\phi_k(x) = e^{ikx} = \cos(kx) + i\sin(kx), \quad k = 0, \pm 1, \pm 2, \dots$$

$$(\phi_j, \phi_k) = \int_{x=0}^{2\pi} \phi_j(x)\phi_k^\star(x)dx = \int_0^{2\pi} \exp\left[i(j-k)x\right] dx = 2\pi\delta_{jk}$$

- The complex exponentials can be shown to form a complete **trigonometric polynomial basis** for the space $L^2_{2\pi}$, i.e.,

$$\forall f \in L^2_{2\pi}: \quad f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ikx},$$

$$\hat{f}_k = \frac{(f, \phi_k)}{2\pi} = \frac{1}{2\pi}. \int_0^{2\pi} f(x) e^{-ikx} dx.$$

Discrete Fourier Basis

$$\forall \Delta k \in \mathbb{Z}: \quad \frac{2\pi}{N} \sum_{j} \exp\left[i \frac{2\pi}{N} j \Delta k \right] = 2\pi \delta_{\Delta k}$$

- The Fourier basis is **discretely orthogonal**

$$\phi_k \cdot \phi_{k'} = \frac{2\pi}{N} \sum_{j=0}^{N-1} (\phi_k)_j \, (\phi_{k'})_j = 2\pi \delta_{k,k'}$$

- This gives the Fourier interpolating polynomial (**spectral approximation**):

$$\text{Forward } \mathbf{f} \rightarrow \hat{\mathbf{f}}: \quad \hat{f}_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j \exp\left(-\frac{2\pi ijk}{N} \right)$$

$$\text{Inverse } \hat{\mathbf{f}} \rightarrow f: \quad f(x) \approx \phi(x) = \sum_{k=-(N-1)/2}^{(N-1)/2} \hat{f}_k e^{ikx}$$

Spectral Convergence (or not)

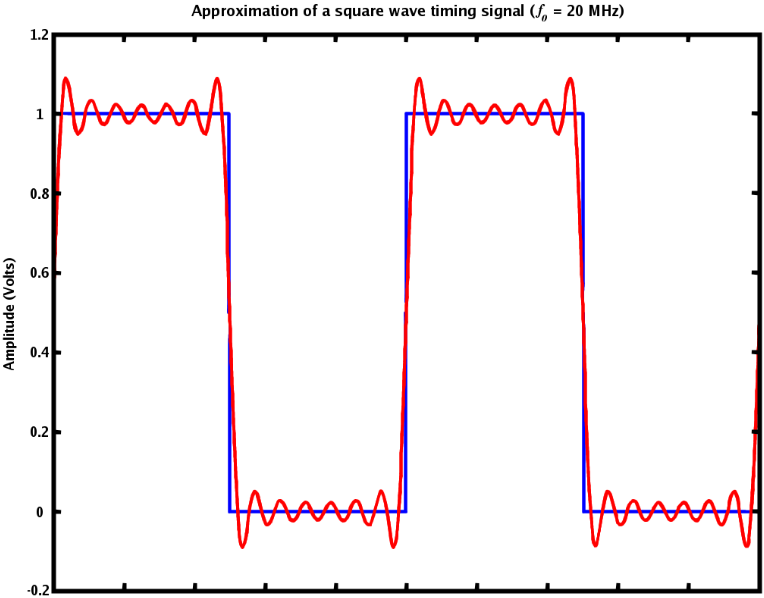
- The Fourier interpolating polynomial $\phi(x)$ has **spectral accuracy**, i.e., exponential in the number of nodes N

$$\|f(x)-\phi(x)\|\sim e^{-N}$$

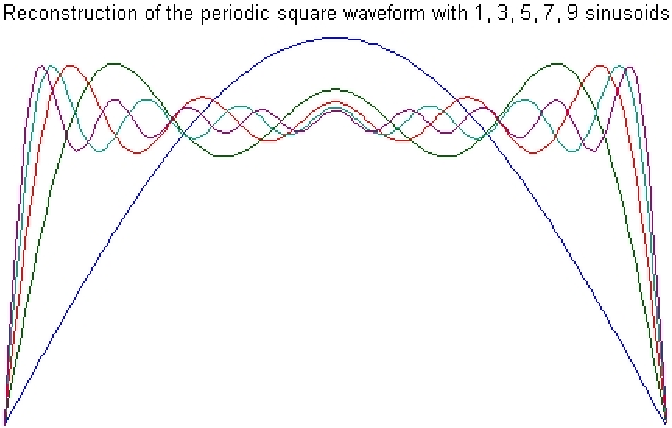
  for **sufficiently smooth functions**.
- Specifically, what is needed is sufficiently **rapid decay of the Fourier coefficients** with k, e.g., exponential decay $\left|\hat{f}_k\right|\sim e^{-|k|}$.
- Discontinuities cause slowly-decaying Fourier coefficients, e.g., power law decay $\left|\hat{f}_k\right|\sim k^{-1}$ for **jump discontinuities**.
- Jump discontinuities lead to slow convergence of the Fourier series for non-singular points (and no convergence at all near the singularity), so-called **Gibbs phenomenon** (ringing):

$$\|f(x)-\phi(x)\|\sim\begin{cases}N^{-1} & \text{at points away from jumps}\\ \text{const.} & \text{at the jumps themselves}\end{cases}$$

Gibbs Phenomenon

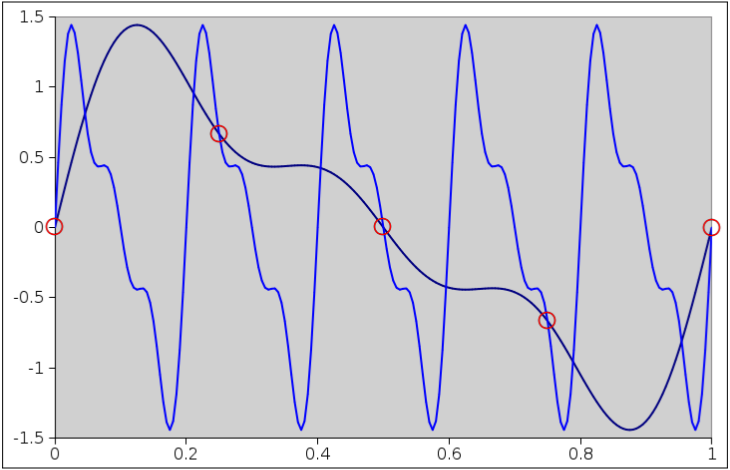


Gibbs Phenomenon



Aliasing

If we sample a signal at too few points the Fourier interpolant may be wildly wrong: **aliasing** of frequencies k and $2k, 3k, \ldots$



DFT

- Recall the transformation from real space to frequency space and back:

$$\mathbf{f} \rightarrow \hat{\mathbf{f}}: \quad \hat{f}_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j \exp\left(-\frac{2\pi ijk}{N}\right), \quad k = -\frac{(N-1)}{2}, \dots, \frac{(N-1)}{2}$$

$$\hat{\mathbf{f}} \rightarrow \mathbf{f}: \quad f_j = \sum_{k=-(N-1)/2}^{(N-1)/2} \hat{f}_k \exp\left(\frac{2\pi ijk}{N}\right), \quad j = 0, \dots, N-1$$

- We can make the forward-reverse **Discrete Fourier Transform** (DFT) more symmetric if we shift the frequencies to $k = 0, \dots, N$:

$$\text{Forward } \mathbf{f} \rightarrow \hat{\mathbf{f}}: \quad \hat{f}_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} f_j \exp\left(-\frac{2\pi ijk}{N}\right), \quad k = 0, \dots, N-1$$

$$\text{Inverse } \hat{\mathbf{f}} \rightarrow \mathbf{f}: \quad f_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{f}_k \exp\left(\frac{2\pi ijk}{N}\right), \quad j = 0, \dots, N-1$$

FFT

- We can write the transforms in matrix notation:

$$\hat{\mathbf{f}} = \frac{1}{\sqrt{N}} \mathbf{U}_N \mathbf{f}$$
$$\mathbf{f} = \frac{1}{\sqrt{N}} \mathbf{U}_N^\star \hat{\mathbf{f}},$$

  where the **unitary Fourier matrix** ($fft(eye(N))$ in MATLAB) is an $N \times N$ matrix with entries

$$u_{jk}^{(N)} = \omega_N^{jk}, \quad \omega_N = e^{-2\pi i/N}.$$

- A **direct** matrix-vector multiplication algorithm therefore takes $O(N^2)$ multiplications and additions.
- Is there a faster way to compute the **non-normalized**

$$\hat{f}_k = \sum_{j=0}^{N-1} f_j \omega_N^{jk} \quad ?$$

FFT

- For now assume that N is even and in fact a power of two, $N = 2^n$.
- The idea is to split the transform into two pieces, **even and odd** points:

$$\sum_{j=2j'} f_j \omega_N^{jk} + \sum_{j=2j'+1} f_j \omega_N^{jk} = \sum_{j'=0}^{N/2-1} f_{2j'} \left(\omega_N^2\right)^{j'k} + \omega_N^k \sum_{j'=0}^{N/2-1} f_{2j'+1} \left(\omega_N^2\right)^{j'k}$$

- Now notice that

$$\omega_N^2 = e^{-4\pi i/N} = e^{-2\pi i/(N/2)} = \omega_{N/2}$$

- This leads to a **divide-and-conquer algorithm**:

$$\hat{f}_k = \sum_{j'=0}^{N/2-1} f_{2j'} \omega_{N/2}^{j'k} + \omega_N^k \sum_{j'=0}^{N/2-1} f_{2j'+1} \omega_{N/2}^{j'k}$$

$$\hat{f}_k = \mathbf{U}_N \mathbf{f} = \left(\mathbf{U}_{N/2} \mathbf{f}_{even} + \omega_N^k \mathbf{U}_{N/2} \mathbf{f}_{odd}\right)$$

FFT Complexity

- The **Fast Fourier Transform** algorithm is recursive:

  $$FFT_N(\mathbf{f}) = FFT_{\frac{N}{2}}(\mathbf{f}_{even}) + \mathbf{w} \boxdot FFT_{\frac{N}{2}}(\mathbf{f}_{odd}),$$

  where $w_k = \omega_N^k$ and $\boxdot$ denotes element-wise product. When $N = 1$ the FFT is trivial (identity).
- To compute the whole transform we need $\log_2(N)$ steps, and at each step we only need N multiplications and $N/2$ additions at each step.
- The total **cost of FFT** is thus much better than the direct method's $O(N^2)$: **Log-linear**

  $$O(N \log N).$$
- Even when N is not a power of two there are ways to do a similar **splitting** transformation of the large FFT into many smaller FFTs.
- Note that there are different **normalization conventions** used in different software.

Applications of FFTs

- Because FFT is a very fast, almost linear algorithm, it is used often to accomplish things that are not seemingly related to function approximation.
- Denote the Discrete Fourier transform, computed using FFTs in practice, with
$$\hat{\mathbf{f}} = \mathcal{F}\left(\mathbf{f}\right) \text{ and } \mathbf{f} = \mathcal{F}^{-1}\left(\hat{\mathbf{f}}\right).$$
- Plain FFT is used in signal processing for **digital filtering**: Multiply the spectrum by a filter $\hat{S}(k)$ discretized as $\hat{\mathbf{s}} = \left\{\hat{S}(k)\right\}_k$:
$$\mathbf{f}_{filt} = \mathcal{F}^{-1}\left(\hat{\mathbf{s}} \boxdot \hat{\mathbf{f}}\right) = \mathbf{f} \circledast \mathbf{s},$$
where $\circledast$ denotes convolution, to be described shortly.
- Examples include **low-pass**, **high-pass**, or **band-pass filters**. Note that **aliasing** can be a problem for digital filters.

Convolution

- For continuous function, an important type of operation found in practice is **convolution** of a (periodic) function $f(x)$ with a (periodic) **kernel** $K(x)$:

$$(K \circledast f)(x) = \int_0^{2\pi} f(y)K(x-y)dy = (f \circledast K)(x).$$

- It is not hard to prove the **convolution theorem**:

$$\mathcal{F}(K \circledast f) = \mathcal{F}(K) \cdot \mathcal{F}(f).$$

- Importantly, this remains true for **discrete convolutions**:

$$(\mathbf{K} \circledast \mathbf{f})_j = \frac{1}{N} \sum_{j'=0}^{N-1} f_{j'} \cdot K_{j-j'} \quad \Rightarrow$$

$$\mathcal{F}(\mathbf{K} \circledast \mathbf{f}) = \mathcal{F}(\mathbf{K}) \cdot \mathcal{F}(\mathbf{f}) \quad \Rightarrow \quad \mathbf{K} \circledast \mathbf{f} = \mathcal{F}^{-1}\left(\mathcal{F}(\mathbf{K}) \cdot \mathcal{F}(\mathbf{f})\right)$$

Proof of Discrete Convolution Theorem

Assume that the normalization used is a factor of N^{-1} in the forward and no factor in the reverse DFT:

$$\mathcal{F}^{-1}\left(\mathcal{F}\left(\mathbf{K}\right)\cdot\mathcal{F}\left(\mathbf{f}\right)\right)=\mathbf{K}\circledast\mathbf{f}$$

$$\left[\mathcal{F}^{-1}\left(\mathcal{F}\left(\mathbf{K}\right)\cdot\mathcal{F}\left(\mathbf{f}\right)\right)\right]_k=\sum_{k=0}^{N-1}\hat{f}_k\hat{K}_k\exp\left(\frac{2\pi ijk}{N}\right)=$$

$$N^{-2}\sum_{k=0}^{N-1}\left(\sum_{l=0}^{N-1}f_l\exp\left(-\frac{2\pi ilk}{N}\right)\right)\left(\sum_{m=0}^{N-1}K_m\exp\left(-\frac{2\pi imk}{N}\right)\right)\exp\left(\frac{2\pi ijk}{N}\right)$$

$$=N^{-2}\sum_{l=0}^{N-1}f_l\sum_{m=0}^{N-1}K_m\sum_{k=0}^{N-1}\exp\left[\frac{2\pi i\left(j-l-m\right)k}{N}\right]$$

contd.

Recall the key discrete orthogonality property

$$\forall \Delta k \in \mathbb{Z} : \quad N^{-1} \sum_{j} \exp\left[i \frac{2\pi}{N} j \Delta k \right] = \delta_{\Delta k} \quad \Rightarrow$$

$$N^{-2} \sum_{l=0}^{N-1} f_l \sum_{m=0}^{N-1} K_m \sum_{k=0}^{N-1} \exp\left[\frac{2\pi i \left(j - l - m \right) k}{N} \right] = N^{-1} \sum_{l=0}^{N-1} f_l \sum_{m=0}^{N-1} K_m \delta_{j-l-m}$$

$$= N^{-1} \sum_{l=0}^{N-1} f_l K_{j-l} = \left(\mathbf{K} \circledast \mathbf{f} \right)_j$$

Computing convolutions requires 2 forward FFTs, one element-wise product, and one inverse FFT, for a total cost $N \log N$ instead of N^2.

Spectral Derivative

- Consider approximating the derivative of a periodic function $f(x)$, computed at a set of N equally-spaced nodes, $\mathbf{f}$.
- One way to do it is to use the **finite difference approximations**:

$$f'(x_j) \approx \frac{f(x_j+h)-f(x_j-h)}{2h} = \frac{f_{j+1}-f_{j-1}}{2h}.$$

- In order to achieve spectral accuracy of the derivative, we can differentiate the spectral approximation: **Spectral derivative**

$$f'(x) \approx \phi'(x) = \frac{d}{dx}\phi(x) = \frac{d}{dx}\left(\sum_{k=0}^{N-1}\hat{f}_k e^{ikx}\right) = \sum_{k=0}^{N-1}\hat{f}_k \frac{d}{dx} e^{ikx}$$

$$\phi' = \sum_{k=0}^{N-1}\left(ik\hat{f}_k\right)e^{ikx} = \mathcal{F}^{-1}\left(i\hat{\mathbf{f}} \boxdot \mathbf{k}\right)$$

- **Differentiation, like convolution, becomes multiplication in Fourier space.**

Multidimensional FFT

- DFTs and FFTs generalize straightforwardly to higher dimensions due to separability: **Transform each dimension independently**

$$\hat{f} = \frac{1}{N_x N_y} \sum_{j_y=0}^{N_y-1} \sum_{j_x=0}^{N_x-1} f_{j_x,j_y} \exp\left[-\frac{2\pi i\left(j_x k_x + j_y k_y\right)}{N}\right]$$

$$\hat{\mathbf{f}}_{k_x,k_y} = \frac{1}{N_x} \sum_{j_y=0}^{N_y-1} \exp\left(-\frac{2\pi i j_y k_x}{N}\right) \left[\frac{1}{N_y} \sum_{j_y=0}^{N_y-1} f_{j_x,j_y} \exp\left(-\frac{2\pi i j_y k_y}{N}\right)\right]$$

- For example, in two dimensions, **do FFTs of each column, then FFTs of each row of the result**:

$$\hat{\mathbf{f}} = \mathcal{F}_{row}\left(\mathcal{F}_{col}\left(\mathbf{f}\right)\right)$$

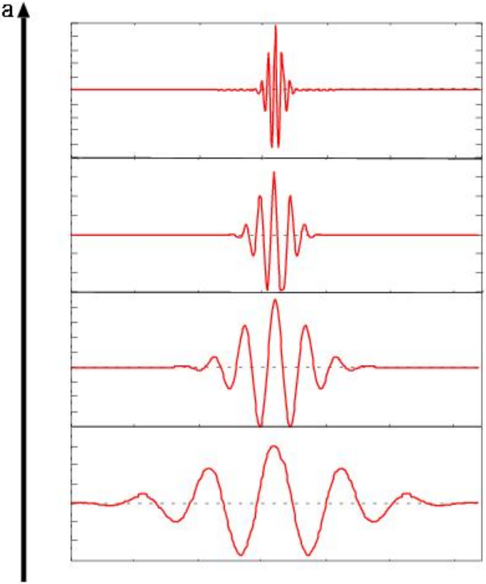
- The cost is N_y one-dimensional FFTs of length N_x and then N_x one-dimensional FFTs of length N_y:

$$N_x N_y \log N_x + N_x N_y \log N_y = N_x N_y \log\left(N_x N_y\right) = N \log N$$

The need for wavelets

- Fourier basis is great for analyzing periodic signals, but is not good for functions that are **localized in space**, e.g., brief bursts of speach.
- Fourier transforms are not good with handling **discontinuities** in functions because of the Gibbs phenomenon.
- Fourier polynomails **assume periodicity** and are not as useful for non-periodic functions.
- Because Fourier basis is not localized, the highest frequency present in the signal must be used everywhere: One cannot use **different resolutions in different regions of space**.

An example wavelet



Wavelet basis

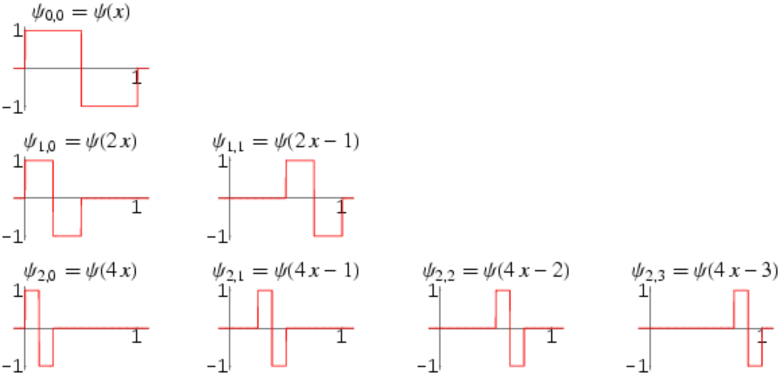
- A **mother wavelet function** $W(x)$ is a localized function in space. For simplicity assume that $W(x)$ has compact support on $[0,1]$.
- A **wavelet basis** is a collection of **wavelets** $W_{s,\tau}(x)$ obtained from $W(x)$ by **dilation** with a **scaling factor** s and **shifting** by a **translation factor** τ:

$$W_{s,\tau}(x) = W\left(sx - \tau\right).$$

- Here the scale plays the role of frequency in the FT, but the shift is novel and localized the basis functions in space.
- We focus on **discrete wavelet basis**, where the scaling factors are chosen to be powers of 2 and the shifts are integers:

$$W_{j,k} = W(2^j x - k), \quad k \in \mathbb{Z},\ j \in \mathbb{Z},\ j \geq 0.$$

Haar Wavelet Basis



Wavelet Transform

- Any function can now be represented in the wavelet basis:

$$f(x) = c_0 + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} c_{jk} W_{j,k}(x)$$

This representation picks out frequency components in different spatial regions.

- As usual, we truncate the basis at $j < J$, which leads to a total number of coefficients c_{jk}:

$$\sum_{j=0}^{J-1} 2^j = 2^J$$

Discrete Wavelet Basis

- Similarly, we discretize the function on a set of $N = 2^J$ equally-spaced nodes $x_{j,k}$ or intervals, to get the vector $\mathbf{f}$:

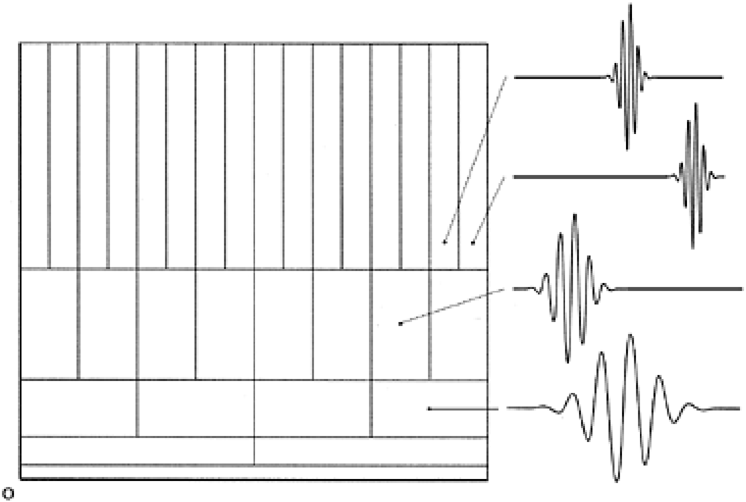
$$\mathbf{f} = c_0 + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} c_{jk} W_{j,k}(x_{j,k}) = \mathbf{W}_j \mathbf{c}$$

- In order to be able to quickly and stably compute the coefficients $\mathbf{c}$ we need an **orthogonal wavelet basis**:

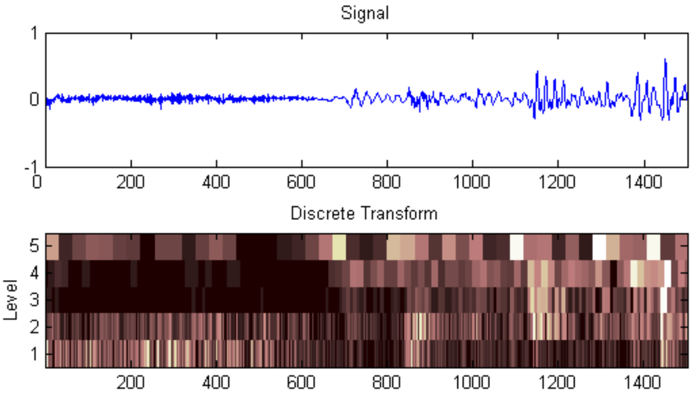
$$\int W_{j,k}(x) W_{l,m}(x) dx = \delta_{j,l} \delta_{l,m}$$

- The Haar basis is discretely orthogonal and computing the transform and its inverse can be done using a **fast wavelet transform**, in **linear time** $O(N)$ time.

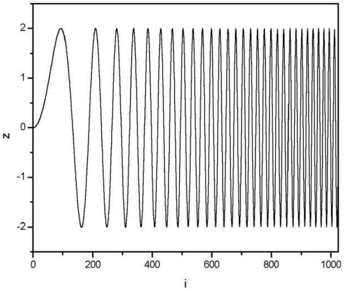
Discrete Wavelet Transform



Scaleogram


Signal
1
0
-1
0
200
400
600
800
1000
1200
1400
Discrete Transform
Level
5
4
3
2
1
200
400
600
800
1000
1200
1400

Another scaleogram



Daubechies Wavelets

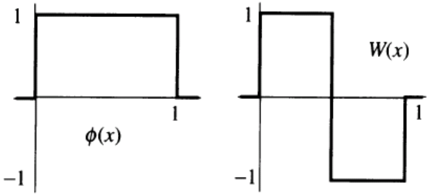
- For the Haar basis, the **wavelet approximation**

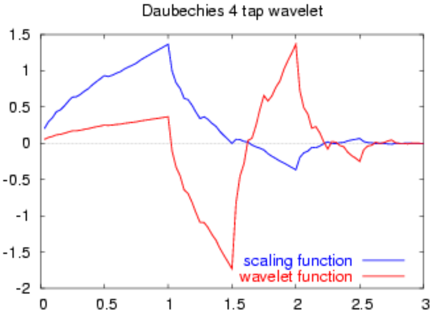
$$\phi(x) = c_0 + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} c_{jk} W_{j,k}(x)$$

  is **piecewise constant** on each of the N sub-intervals of $[0,1]$.
- It is desirable to construct wavelet basis for which:
  - The basis is **orthogonal**.
  - One can exactly represent linear functions (**differentiable**).
  - One can **compute** the forward and reverse wavelet transforms **efficiently**.
- Constructions of such basis start from a **father wavelet function** $\phi(x)$:

$$\phi(x) = \sum_{k=0}^{N} c_k \phi(2x-k), \text{ and } W(x) = \sum_{k=1-N}^{1} (-1)^k c_{1-k} \phi(2x-k)$$

Mother and Father Wavelets





Conclusions/Summary

- **Periodic functions** can be approximated using basis of **orthogonal trigonometric polynomials**.
- The Fourier basis is **discretely orthogonal** and gives **spectral accuracy** for smooth functions.
- Functions with discontinuities are not approximated well: **Gibbs phenomenon**.
- The **Discrete Fourier Transform** can be computed very efficiently using the **Fast Fourier Transform** algorithm: $O(N \log N)$.
- FFTs can be used to **filter** signals, to do **convolutions**, and to provide spectrally-accurate **derivatives**, all in $O(N \log N)$ time.
- For signals that have different properties in different parts of the domain a **wavelet basis** may be more appropriate.
- Using specially-constructed **orthogonal discrete wavelet basis** one can compute **fast discrete wavelet transforms** in time $O(N)$.