Computational methods for suspensions of (cross-linked) slender fibers

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Outline

1. Motivation
2. Fibers in Stokes flow
   - Hydrodynamics
   - Adding twist
   - Inextensibility
3. Numerical Methods
4. Actin gels
5. Adding Brownian motion

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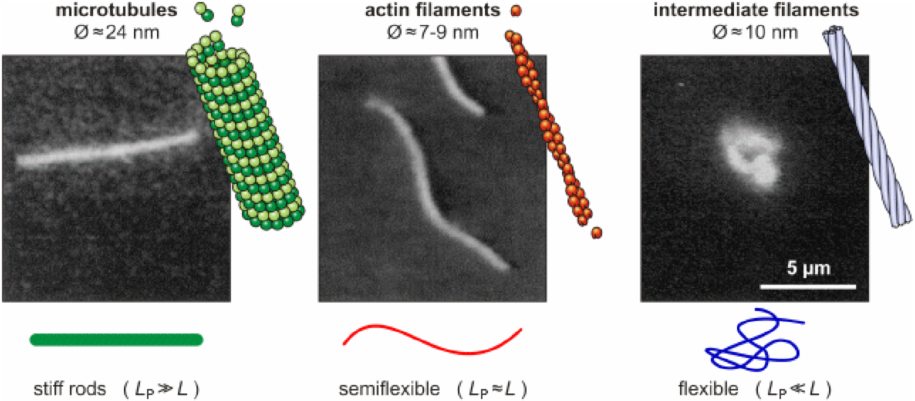
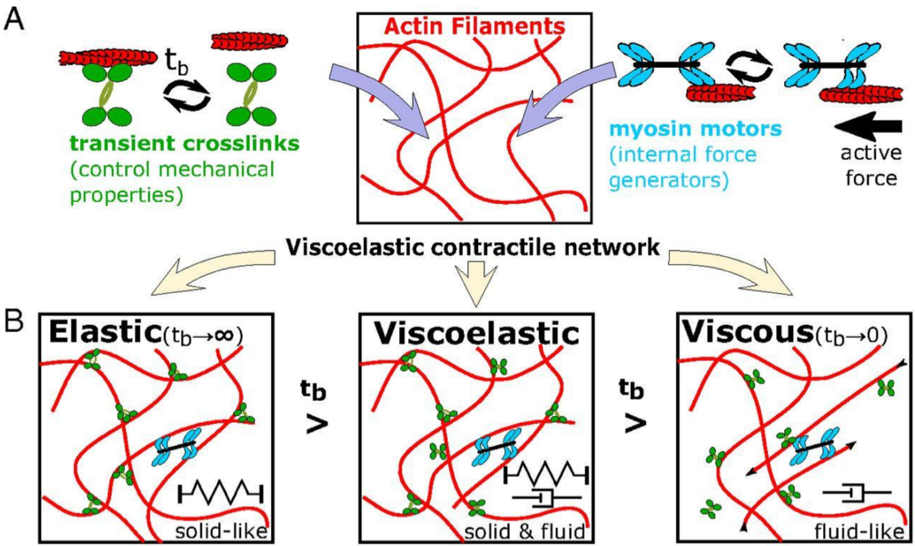
Fibers involved in cell mechanics



Pawlizak and Käs, University of Leipzig

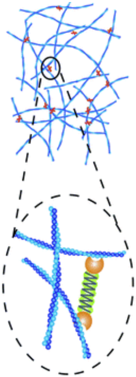
L_p =persistence length, L =fiber length, $a = \epsilon L$ =fiber radius, ϵ =slenderness ratio

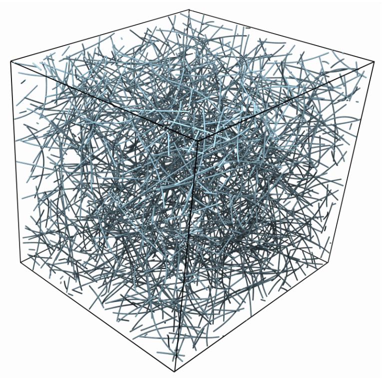
Cytoskeleton rheology



Ahmed and Betz. *PNAS*. (2015)

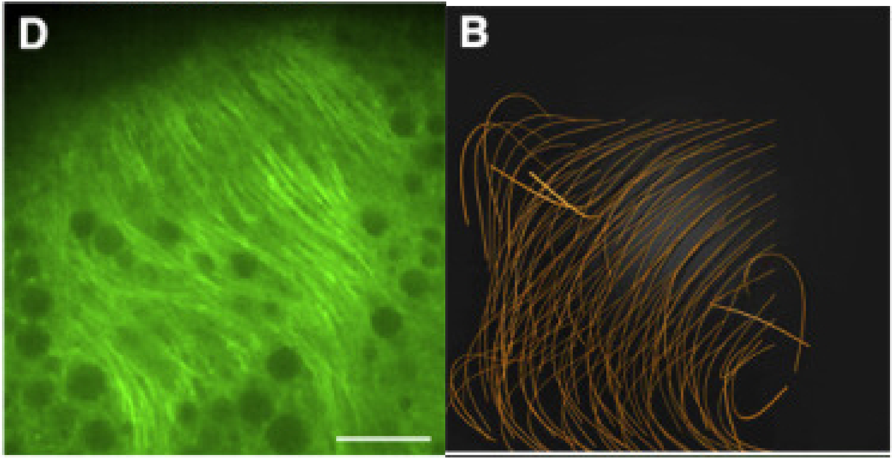
Cross-linked actin gels





- Very **slender semi-flexible fibers** (aspect ratio $10^2 - 10^4$) suspended in a **viscous solvent**.
- For now **cross linkers** modeled as simple elastic springs.
- **Periodic cyclically sheared** unit cell: **viscoelastic moduli**.

Does nonlocal hydrodynamics matter?



Monteith et al. Biophysics Journal. (2016)

Does nonlocal hydrodynamics matter?

- Sometimes flows created by individual fibers add up constructively to produce **large-scale flows**, which advect network.
- For example, cytoplasmic streaming on previous slide or contraction of a myosin-actin gel (must expel liquid out).
- Flow is generated at scales of fiber thickness: **multiscale problem**.
- Role of **long-ranged (nonlocal) hydrodynamics** unclear for rheology of cross-linked actin gels.

Dynamics of Flexible Fibers in Viscous Flows and Fluids, Ann. Rev. Fluid Mech. 51:539, du Roure, Lindner, Nazockdast, Shelley [1]

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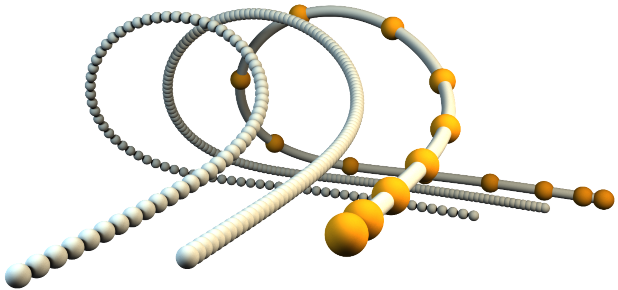
Fiber Representation

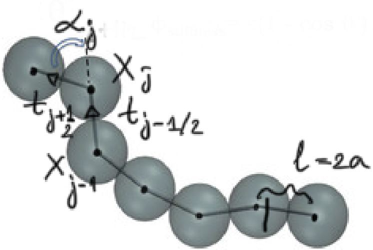
Simple approach is to represent a fiber as a **discrete chain** of beads/blobs: **multiblob model**



More efficient approach is to represent a fibers as **continuum curve**
O. Maxian et al. **ArXiv:2201.04187**

An integral-based spectral method for inextensible slender fibers in Stokes flow [2]
The hydrodynamics of a twisting, bending, inextensible fiber in Stokes flow [3]

Inextensible multiblob chains



Worm-like polymer chain

- Inextensibility: $\|\mathbf{X}_{j+1} - \mathbf{X}_j\| = l \sim a$ (e.g., a or $2a$).
- Tangent vectors: $\boldsymbol{\tau}_{j+1/2} = (\mathbf{X}_{j+1} - \mathbf{X}_j)/l$
- Bending angles: $\cos\alpha_j = \boldsymbol{\tau}_{j+1/2} \cdot \boldsymbol{\tau}_{j-1/2}$
- Elastic energy (bending modulus κ_b)

$$E_b = \frac{2\kappa_b}{l} \sum_{j=1}^{N-1} \sin^2\left(\frac{\alpha_j}{2}\right)$$

Inextensible continuum fibers

- Persistence length due to thermal fluctuations $\xi = 2\kappa_b / (k_B T) \gg l$ gives us a **continuum limit**, $\alpha_j \ll 1$.
- Fiber centerline $\mathbf{X}(s)$ where the **arc length** $0 \leq s \leq L$.
- The tangent vector is $\boldsymbol{\tau} = \partial \mathbf{X} / \partial s = \mathbf{X}_s$, and the **fibers are inextensible**,

$$\boldsymbol{\tau}(s,t) \cdot \boldsymbol{\tau}(s,t) = 1 \quad \forall (s,t).$$

- Bending energy functional is integral of curvature squared:

$$E_b(\mathbf{X}) = \frac{2\kappa_b}{l} \sum_{j=1}^{N-1} \left(\frac{\alpha_j}{2} \right)^2 \quad \Rightarrow \quad E_b[\mathbf{X}(\cdot)] = \frac{\kappa_b}{2} \int ds \, \| \mathbf{X}_{ss}(s) \|^2$$

Bending elasticity

- Bending force $\mathbf{F}_j^{(b)}$ on interior blob j gives us **elastic force density**

$$\mathbf{F}_j^{(b)} = -\frac{\partial E_b}{\partial \mathbf{X}_j} = \frac{\kappa_b}{l^3}\left(-\mathbf{X}_{j-2} + 4\mathbf{X}_{j-1} - 6\mathbf{X}_j + 4\mathbf{X}_{j+1} - \mathbf{X}_{j+2}\right)$$

$$\mathbf{F}_b \approx -l\kappa_b\,\mathbf{D}^4\mathbf{X} \quad \Rightarrow \quad \mathbf{f}_b = -\frac{\delta E_{\text{bend}}}{\delta X} = -\kappa_b \mathbf{X}_{ssss}$$

- Endpoints naturally handled discretely, giving in continuum natural BCs for **free fibers**:

$$\mathbf{X}_{ss}\left(0/L\right) = 0, \quad \mathbf{X}_{sss}\left(0/L\right) = 0.$$

- **Tensions** $T_{j+1/2} \to T(s)$ are **unknown** and resist stretching,

$$\boldsymbol{\Lambda}_i = T_{i+1/2}\boldsymbol{\tau}_{i+1/2} - T_{i-1/2}\boldsymbol{\tau}_{i-1/2} \quad \Rightarrow \quad \boldsymbol{\lambda} = \left(T\boldsymbol{\tau}\right)_s.$$

Fluid dynamics

- For multiblob chains in **Stokes flow**, fluid velocity $\mathbf{v}(\mathbf{r},t)$ satisfies $\nabla\cdot\mathbf{v}=\mathbf{0}$ and
$$\nabla\pi=\eta\nabla^2\mathbf{v}+\sum_j\mathbf{F}_j\,\delta_a\left(\mathbf{X}_j-\mathbf{r}\right),$$
where $\delta_a(\mathbf{r})$ is a **blob kernel** of width $\sim a$, and
$$\mathbf{F}=-l\kappa_b\,\mathbf{D}^4\mathbf{X}+\boldsymbol{\Lambda}$$
- Blobs/fiber are advected by fluid
$$\mathbf{U}_j=d\mathbf{X}_j/dt=\int d\mathbf{r}\;\mathbf{v}(\mathbf{r},t)\,\delta_a\left(\mathbf{X}_j-\mathbf{r}\right).$$
- Continuum limit is obvious (without Brownian fluctuations)
$$\nabla\pi(\mathbf{r},t)=\eta\nabla^2\mathbf{v}(\mathbf{r},t)+\int_0^L ds\,\mathbf{f}(s,t)\delta_a\left(\mathbf{X}(s,t)-\mathbf{r}\right)$$
$$\mathbf{U}(s,t)=\partial_t\mathbf{X}(s,t)=\int d\mathbf{r}\;\mathbf{v}(\mathbf{r},t)\,\delta_a\left(\mathbf{X}(s,t)-\mathbf{r}\right)$$
$$\mathbf{f}=-\kappa_b\mathbf{X}_{ssss}+\boldsymbol{\lambda}$$

Multiblob chains in Stokes flow

- We can (temporarily) eliminate the fluid velocity to write an equation for **fiber only**.
- Define the positive semi-definite **hydrodynamic kernel**

$$\mathcal{R}\left(\mathbf{r}_1, \mathbf{r}_2\right) = \int \delta_a\left(\mathbf{r}_1 - \mathbf{r}'\right) \mathbb{G}\left(\mathbf{r}', \mathbf{r}''\right) \delta_a\left(\mathbf{r}_2 - \mathbf{r}''\right) d\mathbf{r}' d\mathbf{r}'',$$

  where $\mathbb{G}$ is the Green's function for (periodic) Stokes flow.
- Define $\mathbf{M}\left(\mathbf{X}\right) \succeq \mathbf{0}$ to be the symmetric positive semidefinite (SPD) **mobility matrix** with blocks

$$\mathbf{M}_{ij}\left(\mathbf{X}_i, \mathbf{X}_j\right) = \mathcal{R}\left(\mathbf{X}_i, \mathbf{X}_j\right) = \mathcal{R}\left(\mathbf{X}_i - \mathbf{X}_j\right).$$

- Discrete dynamics = **inextensibility** +

$$\mathbf{U} = d\mathbf{X}/dt = \mathbf{M}\left(\mathbf{X}\right) \mathbf{F}\left(\mathbf{X}\right) = \mathbf{M}\left(-l\kappa_b \mathbf{D}^4\mathbf{X} + \boldsymbol{\Lambda}\right)$$

Inextensible fibers in Stokes flow

- Define a positive semidefinite **mobility operator**

$$(\mathcal{M}[\mathbf{X}(\cdot)]\,\mathbf{f}(\cdot))(s) = \int_0^L ds'\,\mathcal{R}\left(\mathbf{X}(s),\mathbf{X}(s')\right)\mathbf{f}(s')$$

- Continuum dynamics is a **non-local PDE**

$$\mathbf{U} = \mathbf{X}_t = \mathcal{M}[\mathbf{X}]\left(-\kappa_b \mathbf{X}_{ssss} + \boldsymbol{\lambda}\right)$$

$$\boldsymbol{\tau}(s,t)\cdot\boldsymbol{\tau}(s,t) = 1 \quad \forall (s,t).$$

- Is this PDE well-posed? We have shown *numerically* that
  - Fiber velocity converges pointwise (strongly) up to the endpoints.
  - **Moments of $\boldsymbol{\lambda}$ converge**, e.g., stress tensor (weak convergence).

Rotne-Prager-Yamakawa kernel

$$\mathcal{R}\left(\mathbf{r}_1,\mathbf{r}_2\right)=\int\delta_a\left(\mathbf{r}_1-\mathbf{r}'\right)\mathbb{G}\left(\mathbf{r}',\mathbf{r}''\right)\delta_a\left(\mathbf{r}_2-\mathbf{r}''\right)d\mathbf{r}'d\mathbf{r}''$$

- Taking the regularization kernel and unbounded Stokes flow

$$\delta_a\left(\mathbf{r}\right)=\left(4\pi a^2\right)^{-1}\delta\left(r-a\right)$$

gives the **Rotne-Prager-Yamakawa (RPY) kernel**

$$\mathcal{R}\left(\mathbf{r}\right)=\begin{cases}\left(8\pi\eta\right)^{-1}\left(\boldsymbol{\mathcal{S}}\left(\mathbf{r}\right)+\dfrac{2a^2}{3}\boldsymbol{\mathcal{D}}\left(\mathbf{r}\right)\right), & r>2a\\ \left(6\pi a\eta\right)^{-1}\left[\left(1-\dfrac{9r}{32a}\right)\mathbf{I}+\left(\dfrac{3r}{32a}\right)\dfrac{\mathbf{r}\otimes\mathbf{r}}{r^2}\right], & r\leq 2a\end{cases}$$

$$\boldsymbol{\mathcal{S}}\left(\mathbf{r}\right)=\frac{1}{8\pi\eta r}\left(\mathbf{I}+\hat{\mathbf{r}}\hat{\mathbf{r}}^T\right)\equiv\mathbb{G},\quad\text{and}\quad\boldsymbol{\mathcal{D}}\left(\mathbf{r}\right)=\frac{1}{8\pi\eta r^3}\left(\mathbf{I}-\hat{\mathbf{r}}\hat{\mathbf{r}}^T\right)$$

Slender Body Theory

$$(\mathcal{M}\left[\mathbf{X}\left(\cdot\right)\right]\mathbf{f}\left(\cdot\right))(s)=\int_0^L ds'\,\mathcal{R}\left(\mathbf{X}(s)-\mathbf{X}(s')\right)\mathbf{f}(s')$$

- **Matched asymptotics** gives (away from endpoints)

$$(\mathcal{M}\,\mathbf{f})(s)\approx(\mathcal{M}_{\mathrm{SBT}}\,\mathbf{f})(s)=(\mathcal{M}_{\mathrm{L}}\,\mathbf{f})(s)+(\mathcal{M}_{\mathrm{NL}}\,\mathbf{f})(s)=$$

$$=\frac{1}{8\pi\eta}\left(\log\left(\frac{(L-s)s}{4a^2}\right)\left(\mathbf{I}+\boldsymbol{\tau}(s)\boldsymbol{\tau}(s)^T\right)+4\mathbf{I}\right)\mathbf{f}(s)$$

$$+\frac{1}{8\pi\eta}\int_0^L ds'\left(\mathcal{S}\left(\mathbf{X}(s)-\mathbf{X}(s')\right)\mathbf{f}\left(s'\right)-\left(\frac{\mathbf{I}+\boldsymbol{\tau}(s)\boldsymbol{\tau}(s)^T}{|s-s'|}\right)\mathbf{f}(s)\right)$$

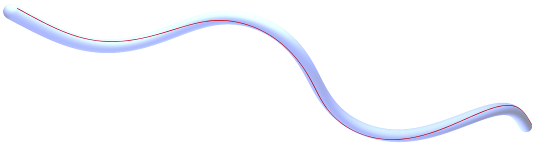
- For a special choice of blob radius $a=\left(e^{3/2}/4\right)\epsilon L=1.12\epsilon L$, this formula matches the widely-used **Slender Body Theory** (SBT).
- Our approach automatically works for **multiple fibers**, and also gives us a natural **regularization of the endpoints** (not shown).

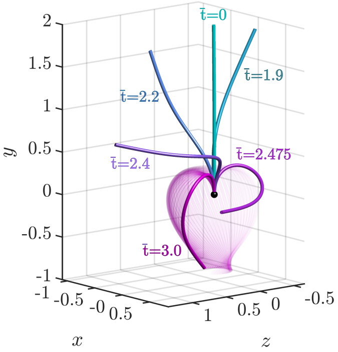
Slender body theory

$$\mathcal{M}_{\text{SBT}} = \mathcal{M}_{\text{L}} + \mathcal{M}_{\text{NL}} = \mathcal{O}\left(\log\left(\frac{(L-s)s}{a^2}\right)\right) + \mathcal{O}(1)$$

- SBT is great for numerics since it involves **quadratures** that can be computed accurately for smooth **f** to **spectral accuracy** (starting with Tornberg+Shelley = TS).
- The local drag term is logarithmically **singular at endpoints** for cylindrical fibers.
  TS use (unphysical) **ellipsoidal fibers**: $\mathcal{M}_{\text{L}} = \mathcal{O}\left(\log\left(L/a\right)\right)$.
- $\mathcal{M}_{\text{L}}$ has **spurious negative eigenvalues** for high spatial frequencies, so $\mathcal{M}_{\text{SBT}}$ is **not SPD** and equations are definitely **not well-posed**.
  TS use **artificial regularization**.

Twisting Fibers





- How to represent twist (Bishop frame)?
- Hydrodynamics with twist? (no slender-body theory exists)
- (When) does twist matter?
  Flagella, formins twisting growing actin filaments, macroscopic chirality in cells, and ?

Twist

- For given force densities $\mathbf{f}(s,t)$ and **parallel torque** densities $n(s,t)$ along the fiber centerlines,

$$\nabla\pi = \eta\nabla^2\mathbf{v} + \int_0^L ds\left[\mathbf{f}(s) + n(s)\frac{\nabla}{2}\times\boldsymbol{\tau}(s)\right]\delta_a\left(\mathbf{X}(s)-\mathbf{r}\right),$$

$$\Omega^{\parallel}(s) = \boldsymbol{\tau}(s)\cdot\int d\mathbf{r}\,\frac{\nabla}{2}\times\mathbf{v}\left(\mathbf{r},t\right)\delta_a\left(\mathbf{X}(s)-\mathbf{r}\right)$$

- Should fiber exert **perpendicular torques** on the fluid? *Not for sufficiently slender fibers* (**ArXiv:2201.04187**) [3].

Bishop frame

- To each point along the fiber we attach an orthonormal triad $\mathbf{B}(s) = [\boldsymbol{\tau}(s), \mathbf{a}(s), \mathbf{b}(s)]$ called the **Bishop frame**, which satisfies the *no-twist condition*:

$$\mathbf{a}_s \cdot \mathbf{b} = 0 \quad \Rightarrow \quad \partial_s \mathbf{a} = (\boldsymbol{\tau} \times \boldsymbol{\tau}_s) \times \mathbf{a}$$

- Configuration represented by **twist angle** $\theta(s)$ between the material frame of the fiber cross section and the Bishop cross section.
- Elastic force has bend-twist coupling (belt trick):

$$\mathbf{f} = -\kappa_b \mathbf{X}_{ssss} + \kappa_t \left(\theta_s \left(\boldsymbol{\tau} \times \boldsymbol{\tau}_s\right)\right)_s + \boldsymbol{\lambda},$$

$$n = \kappa_t \theta_{ss}.$$

- Evolve **twist density** in time via

$$\partial_t \theta_s \left(s, t\right) = \partial_s \Omega^{\parallel} - \left(\boldsymbol{\Omega}^{\perp} \cdot \boldsymbol{\tau}_s\right).$$

Tension equation

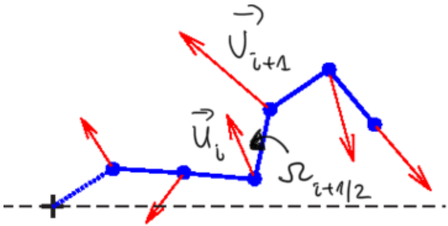
$$\mathbf{X}_t = \mathcal{M}[\mathbf{X}]\left(-\kappa_b \mathbf{X}_{ssss} + \boldsymbol{\lambda}\right) \quad \text{and} \quad \boldsymbol{\lambda} = \left(T\boldsymbol{\tau}\right)_s$$

- Traditional approach (Tornberg+Shelley) is to solve **tension equation**

$$\boldsymbol{\tau} \cdot \boldsymbol{\tau} = \mathbf{X}_s \cdot \mathbf{X}_s = 1 \quad \Rightarrow \quad \left(\mathbf{X}_t\right)_s \cdot \mathbf{X}_s = 0 \quad \text{non-local BVP}$$

- Tension equation is linear in $T(s)$ but very nonlinear in $\mathbf{X}$ and its derivatives, causing **aliasing issues**.
- Method does not strictly enforce inextensibility numerically, requiring adding a **penalty for stretching**.
- To solve these problems, let us first go back to multiblobs for simplicity, and then take a **continuum limit**.

Inextensible motions



$$\frac{\mathbf{U}_i - \mathbf{U}_{i-1}}{\Delta s} = \boldsymbol{\Omega}_{j+1/2} \times \boldsymbol{\tau}_{j+1/2} \quad \Rightarrow$$

$$\mathbf{U} = \mathbf{K}\boldsymbol{\Omega}^{\perp} = \left[\mathbf{U}_0, \cdots, \mathbf{U}_0 + \Delta s \sum_{j=0}^{i-1} \boldsymbol{\Omega}^{\perp}_{j+1/2} \times \boldsymbol{\tau}_{j+1/2}, \cdots \right] \rightarrow$$

$$\left(\mathcal{K}\left[\mathbf{X}\left(\cdot\right)\right]\boldsymbol{\Omega}^{\perp}\left(\cdot\right)\right)(s) = \mathbf{U}\left(s\right) = \mathbf{U}\left(0\right) + \int_0^s ds' \left(\boldsymbol{\Omega}^{\perp}\left(s'\right) \times \boldsymbol{\tau}\left(s'\right)\right).$$

Principle of virtual work

- **Principle of virtual work**: Constraint forces should do no work for any inextensible motion of the fiber:

$$\mathbf{\Lambda}^T \mathbf{U} = \left(\mathbf{K}^T \mathbf{\Lambda}\right)^T \mathbf{\Omega}^{\perp} = 0 \quad \forall \mathbf{\Omega}^{\perp} \quad \Rightarrow \quad \mathbf{K}^T \mathbf{\Lambda} = \mathbf{0}$$

$$\mathbf{K}^T \mathbf{\Lambda} = \left[\sum_{j=0}^{N} \mathbf{\Lambda}_j, \cdots, \Delta s \left(\sum_{j=i}^{N} \mathbf{\Lambda}_j \right) \times \boldsymbol{\tau}_{i+1/2}, \cdots \right] \rightarrow$$

$$\left(\mathcal{K}^{\star}\left[\mathbf{X}\left(\cdot\right)\right] \boldsymbol{\lambda}\left(\cdot\right)\right)(s) = \left[\int_0^L ds' \, \boldsymbol{\lambda}\left(s'\right), \forall s \left(\int_s^L ds' \, \boldsymbol{\lambda}\left(s'\right) \right) \times \boldsymbol{\tau}(s) \right] = 0.$$

- We can express this in terms of tension

$$\forall s \quad \int_s^L ds' \, \boldsymbol{\lambda}\left(s'\right) = -T(s)\boldsymbol{\tau}(s) \quad \Rightarrow \quad \boldsymbol{\lambda} = \left(T\boldsymbol{\tau}\right)_s$$

but the principle of virtual work is an **integral constraint**.

Continuum equations

- New **weak formulation of inextensibility** constraint:

$$\mathbf{X}_t = \mathcal{K}\left[\mathbf{X}\right]\boldsymbol{\Omega}^{\perp} = \mathcal{M}\left[\mathbf{X}\right]\left(-\kappa_b \mathbf{X}_{ssss} + \boldsymbol{\lambda}\right)$$

$$\mathcal{K}^{\star}\left[\mathbf{X}\right]\boldsymbol{\lambda} = \mathbf{0}$$

$$\partial_t \boldsymbol{\tau} = \boldsymbol{\Omega}^{\perp} \times \boldsymbol{\tau}$$

$$\mathbf{X}(s,t) = \mathbf{X}(0,t) + \int_0^s ds' \, \boldsymbol{\tau}\left(ds',\, t\right)$$

- Two improvements:
  - Evolve tangent vector $\boldsymbol{\tau}$ rather than $\mathbf{X}$: **strictly inextensible**.
  - Expose **saddle-point structure** of problem (energy conservation).

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Slender-Body Quadrature

- Recall slender body *theory* (SBT) $\mathcal{M}_{\text{SBT}} = \mathcal{M}_{\text{L}} + \mathcal{M}_{\text{NL}}$.
- We avoid SBT via **slender body quadrature** for RPY

$$\mathbf{U}(s) = \int_{D(s):|s-s'|>2a} \left(\mathbb{S}\left(\mathbf{X}(s), \mathbf{X}(s')\right) + \frac{2a^2}{3} \mathbb{D}\left(\mathbf{X}(s), \mathbf{X}(s')\right) \right) \mathbf{f}\left(s'\right) ds'$$

$$+ \int_{s-2a}^{s+2a} (\ldots \text{RPY} \ldots) \, \mathbf{f}(s') \, ds'.$$

- Apply **singularity subtraction** even though not technically singular:

$$\int_{D(s)} \mathbb{S}\left(\mathbf{X}(s), \mathbf{X}(s')\right) \mathbf{f}\left(s'\right) \, ds' = \frac{1}{8\pi\eta} \int_{D(s)} \left(\frac{\mathbf{I} + \boldsymbol{\tau}(s)\boldsymbol{\tau}(s)}{|s - s'|} \right) \mathbf{f}(s) \, ds'$$

$$+ \int_{D(s)} \left(\mathbb{S}\left(\mathbf{X}(s), \mathbf{X}(s')\right) \mathbf{f}\left(s'\right) - \frac{1}{8\pi\eta} \left(\frac{\mathbf{I} + \boldsymbol{\tau}(s)\boldsymbol{\tau}(s)}{|s - s'|} \right) \mathbf{f}(s) \right) \, ds'$$

- Taking the domain $D(s)$ to be $[0, L]$ in the second gives the finite part integral from SBT!

Spatial Discretization

- We develop a **spectral discretization** in space, based on representing all functions using **Chebyshev polynomials**, with **anti-aliasing**.
- **Collocation discretization** of mobility equation gives a **saddle-point system**

$$\begin{pmatrix} -\mathbf{M}(\mathbf{X}) & \mathbf{K}(\mathbf{X}) \\ \mathbf{K}^*(\mathbf{X}) & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\Omega} \end{pmatrix} = \begin{pmatrix} \mathbf{M}(\mathbf{X})(-\kappa_b \mathbf{D}_{BC}^4 \mathbf{X}) \\ \mathbf{0} \end{pmatrix}$$

  which we solve iteratively using a **block-diagonal preconditioner**.
- We only use $O(16-32)$ Chebyshev points per fiber so doing **dense LA for individual fibers** is OK.
- Bending elasticity can either be discretized using **rectangular collocation** (more accurate, needs BCs) or by discretizing bending energy functional (more robust, **natural BCs**).

Temporal discretization

- **Backward Euler** is the most stable since it ensures strict energy dissipation; also for *dense* suspensions.
- **Split** mobility into **local** (e.g., intra-fiber) and **non-local** (e.g., inter-fiber) parts, $\mathbf{M} = \mathbf{M}_L + \mathbf{M}_{NL}$:

$$\begin{aligned}\mathbf{K}^n \boldsymbol{\Omega}^n =& \mathbf{M}_L^n \left(-\kappa_b \mathbf{D}_{BC}^4 \mathbf{X}^{n+1,\star} + \boldsymbol{\lambda}^{n+1}\right) \\ &+ \mathbf{M}_{NL}^n \left(-\kappa_b \mathbf{D}_{BC}^4 \mathbf{X}^n + \boldsymbol{\lambda}^n\right) + \mathbf{M}\mathbf{f}^n \\ \left(\mathbf{K}^\star\right)^n \boldsymbol{\lambda}^{n+1} =& \mathbf{0},\end{aligned}$$

where $\mathbf{X}^{n+1,\star} = \mathbf{X}^n + \Delta t \mathbf{K}^{n+1/2,*} \boldsymbol{\Omega}^{n+1/2}$.

- Actual fiber update is **strictly inextensible**

$$\boldsymbol{\tau}^{n+1} = \text{rotate}\left(\boldsymbol{\tau}^n, \Delta t \boldsymbol{\Omega}^n\right).$$

- $\mathbf{f}^n$ contains other forces such as **cross-linkers** (can be stiff). **Flow** is easy to add to the rhs.

The gory details

1. For dense suspensions, supplement L+NL splitting with additional **1-5 GMRES iterations** for stability.
2. Evaluate long-ranged hydrodynamic interactions between Chebyshev nodes in **linear time** using *Positively Split Ewald* (PSE) method (FFT based for triply periodic), also works for **deformed/sheared unit cell** (Fiore *et al. J. Chem. Phys.* (2017)).
3. For **nearby fibers**, use specialized **near-singular quadrature** (af Klinteberg and Barnett. BIT Num. Math. 2020 [4]) to get 2-3 digits.
4. For intra-fiber hydro use specialized **slender-body quadrature** ala Anna Karin-Tornberg.

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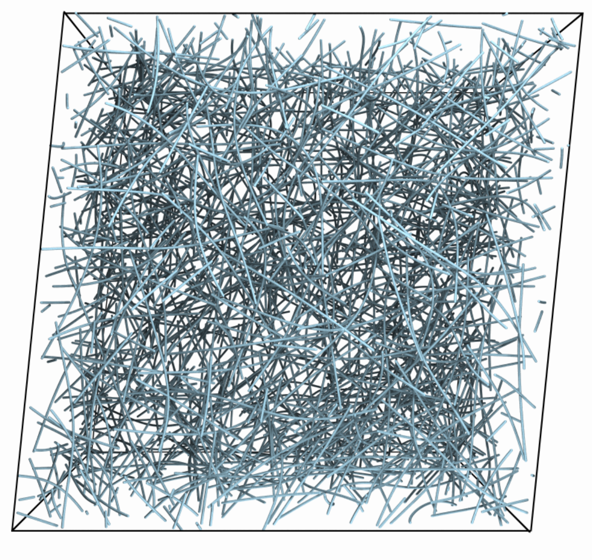
Actin network/gel



Cross Linkers

- Cross linker (CL) between $\mathbf{X}^{(i)}(s_i^*)$ and $\mathbf{X}^{(j)}(s_j^*)$, with $R = \left\| \mathbf{X}^{(i)}(s_i^*) - \mathbf{X}^{(j)}(s_j^*) \right\|$
- Model cross-linker as just a spring with **Gaussian smoothing** to preserve spectral accuracy (std$= \sigma \sim 0.1L$):

$$\mathbf{f}^{(\mathrm{CL},i)}(s) = -K_c \left(1 - \frac{\ell}{R}\right) \delta_\sigma(s - s_i^*) \int_0^L ds' \left(\mathbf{X}^{(i)}(s) - \mathbf{X}^{(j)}(s')\right) \delta_\sigma(s' - s_j^*)$$

- Cross linker is **force and torque-free**.
- Randomly generated dense network of CLs (16 attachment sites per site) to give about 12 CLs per fiber (elastic network).

Cross-linked network

Rheology

Apply linear shear flow $\mathbf{v}_0(x, y, z) = \dot{\gamma}_0 \cos(\omega t) y$ and measure the **visco-elastic stress** induced by the fibers and cross links:

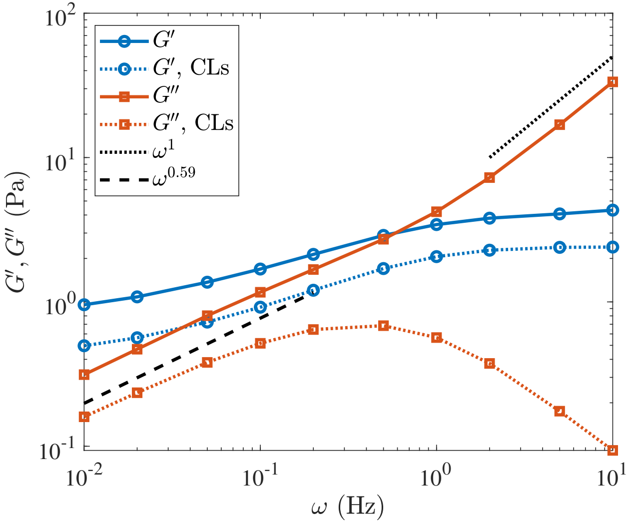
$$\boldsymbol{\sigma}^{(i)} = \frac{1}{V} \sum_{\text{fibers}} \int_0^L ds\, \mathbf{X}^i(s) \left(\mathbf{f}_b(s) + \boldsymbol{\lambda}(s)\right)^T$$

$$\boldsymbol{\sigma}^{(\text{CL})} = \frac{1}{V} \sum_{\text{CLs}=(i,j)} \int_0^L ds \, \left(\mathbf{X}^i(s) \mathbf{f}^{(CL,i)}(s) + \mathbf{X}^j(s) \mathbf{f}^{(CL,j)}(s) \right)$$

$$\frac{\sigma_{21}}{\gamma_0} = G' \sin(\omega t) + G'' \cos(\omega t) = \text{elastic+viscous}.$$

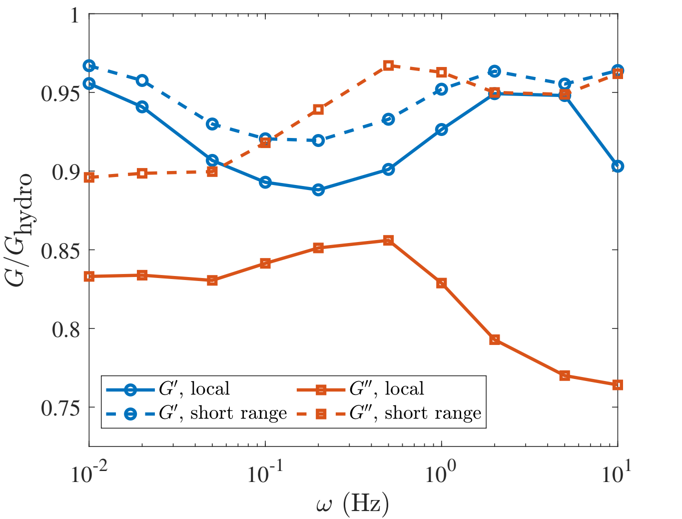
$$G' = \frac{2}{\gamma_0 T} \int_0^T \sigma_{21} \sin(\omega t)\, dt \qquad G'' = \frac{2}{\gamma_0 T} \int_0^T \sigma_{21} \cos(\omega t)\, dt.$$

Viscoelastic moduli: Maxwell fluid



Elastic modulus G' and **viscous** modulus G'' for 700 fibers + 8400 CLs

Nonlocal hydrodynamics



Reduction in viscoelastic moduli with **only local drag** or **only inter-fiber nonlocal hydrodynamics**.

Rheology permanent CLs

- Network relaxation time $\tau_c \approx 0.5 - 1s$
- For $\omega^{-1} \gg \tau_c$
  - Quasi-steady; elastic solid
  - Small effect of nonlocal hydrodynamics ($\sim 10\%$)
- For $\omega^{-1} \approx \tau_c$.
  - $G'' \approx G$
  - Max change in G' due to *inter-fiber* hydro
- For $\omega^{-1} \ll \tau_c$.
  - Fibers and CLs "frozen"; network behaves like a viscous fluid
  - $G'' \gg G'$; up to 25% change due to *intra-fiber* hydro.

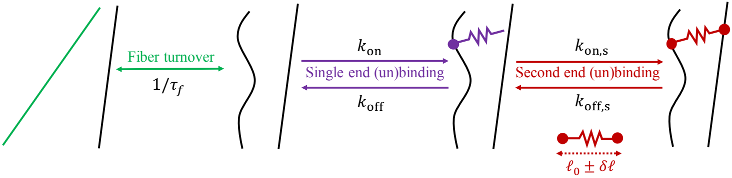
Dynamic cross linking

Kinetic Monte Carlo algorithm for cross linking:

- Discrete set of binding sites on each fiber (for efficiency).
- Doubly-bound CLs act as simple **elastic springs**.

Assumptions behind linking algorithm

- Diffusion of cross-linkers is fast (**diffusion-limited binding**)
- Four reactions between fibers and CL reservoir obey **detailed balance**



"Simulations of dynamically cross-linked actin networks...," O. Maxian et al, PLOS Comp. Bio., 17(12): e1009240, 2021 [**bioRxiv:2021.07.07.451453**] [5]

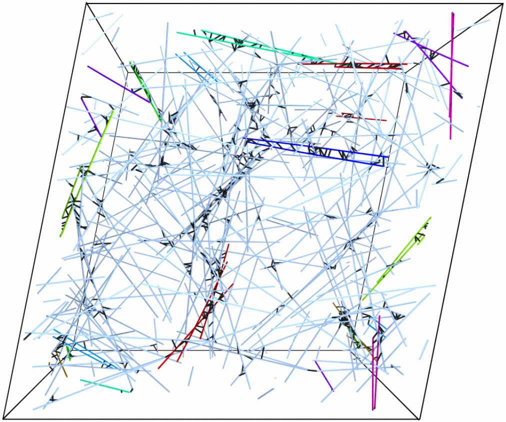
Temporal integrator

We use a **time splitting approach**:

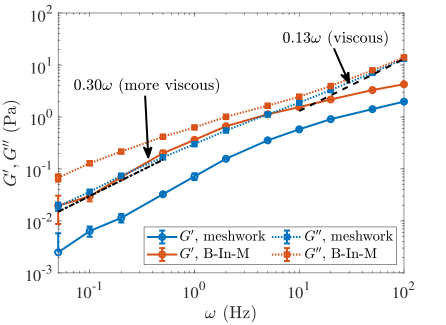
1. Turnover filaments over time Δt (rarely happens).
2. Update cross linkers over time Δt.
3. Calculate $\mathbf{f}^{(CL)}(\mathbf{X})$ and solve
$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{M}(\mathbf{X}) \left[\mathbf{f}^{(\kappa)}(\mathbf{X}) + \mathbf{f}^{(CL)}(\mathbf{X}) + \boldsymbol{\lambda} \right]$$
and update $\mathbf{X}$ over time Δt.
4. Translational and rotational **diffusion of rigid filaments** over time Δt (sometimes).

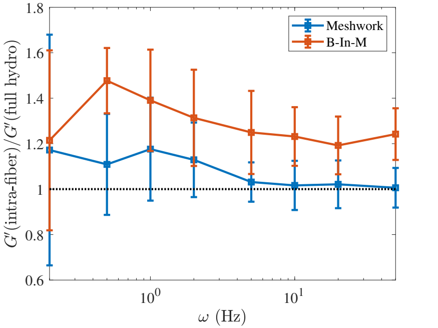
"Interplay between Brownian motion and cross-linking kinetics controls bundling dynamics in actin networks" by O. Maxian et al, in press Biophysical J., 2022 [**bioRxiv:021.09.17.460819**] [6]

Dynamically cross-linked network



Rheology transient CLs





- Measured viscoelastic moduli of dynamically cross-linked networks **without** Brownian motion.
- For bundled networks, elastic modulus overestimated by $\approx 50\%$ without inter-fiber hydro, esp. long timescales.
- Fibers in bundles closer together: stress is reduced because **entrainment flows in bundle** make straining easier.

Outline

1 Motivation

2 Fibers in Stokes flow
- Hydrodynamics
- Adding twist
- Inextensibility

3 Numerical Methods

4 Actin gels

5 Adding Brownian motion

Thermal fluctuations (Brownian Motion)

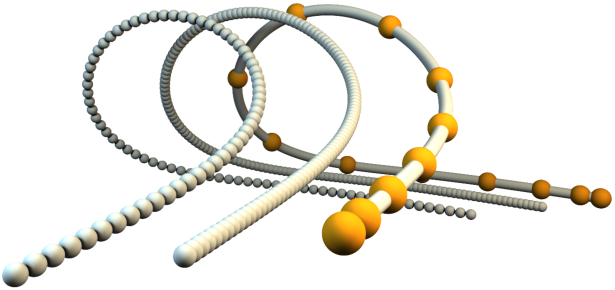
- **Rigid fibers** are "easy" [7] though so far we have only implemented *without* inter-fiber hydro [6].
- Fluctuating hydrodynamics gives the fluctuating Stokes equations

$$\rho\partial_t \mathbf{v} + \nabla\pi = \eta\nabla^2\mathbf{v} + \nabla\cdot\left(\sqrt{2\eta k_B T}\,\mathcal{W}\right) + \int_0^L ds\,\mathbf{f}(s,t)\delta_a\left(\mathbf{X}(s,t)-\mathbf{r}\right).$$

- The **thermal fluctuations** (Brownian motion of fiber) are driven by a white-noise **stochastic stress tensor** $\mathcal{W}(\mathbf{r}, t)$.
- Must first answer deep mathematical questions:
  - Can one make sense of the (multiplicative noise) **overdamped SPDE** for a Brownian curve?
  - Does the **Brownian stress** of the fiber converge in the continuum limit? (bending energy does not)

Brownian multiblob chains

For **Brownian blob-link chains** there are no mathematical issues so start there!



Fast constrained BD-HI for blob-link chains based on rotating unit link vectors including Brownian stress (Brennan Sprinkle, in progress)

References

Olivia Du Roure, Anke Lindner, Ehssan N Nazockdast, and Michael J Shelley.
Dynamics of flexible fibers in viscous flows and fluids.
Annual Review of Fluid Mechanics, 51:539–572, 2019.

Ondrej Maxian, Alex Mogilner, and Aleksandar Donev.
Integral-based spectral method for inextensible slender fibers in stokes flow.
Phys. Rev. Fluids, 6:014102, 2021.

Ondrej Maxian, Brennan Sprinkle, Charles S. Peskin, and Aleksandar Donev.
The hydrodynamics of a twisting, bending, inextensible fiber in stokes flow.
Submitted to Phys. Rev. Fluids, ArXiv:2201.04187, 2022.

Ludvig af Klinteberg and Alex H Barnett.
Accurate quadrature of nearly singular line integrals in two and three dimensions by singularity swapping.
BIT Numerical Mathematics, pages 1–36, 2020.

Ondrej Maxian, Raul Perez Peláez, Alex Mogilner, and Aleksandar Donev.
Simulations of dynamically cross-linked actin networks: Morphology, rheology, and hydrodynamic interactions.
PLOS Computational Biology, 17(12):e1009240, 2021.

Ondrej Maxian, Aleksandar Donev, and Alex Mogilner.
Interplay between brownian motion and cross-linking kinetics controls bundling dynamics in actin networks.
To appear in Biophysics J., bioRxiv:2021.09.17.460819, 2022.

Brennan Sprinkle, Florencio Balboa Usabiaga, Neelesh A. Patankar, and Aleksandar Donev.
Large scale Brownian dynamics of confined suspensions of rigid particles.
J. Chem. Phys., 147(24):244103, 2017.
Software available at `https://github.com/stochasticHydroTools/RigidMultiblobsWall`.